

# WEIL-PETERSSON TEICHMÜLLER SPACE

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**ABSTRACT.** The paper presents some recent results on the Weil-Petersson geometry theory of the universal Teichmüller space, a topic which has wide applications to various areas such as mathematical physics and differential equation. It is shown that a sense-preserving homeomorphism  $h$  on the unit circle belongs to the Weil-Petersson class, namely,  $h$  can be extended to a quasiconformal mapping to the unit disk whose Beltrami coefficient is square integrable in the Poincaré metric if and only if  $h$  is absolutely continuous such that  $\log h'$  belongs to the Sobolev class  $H^{\frac{1}{2}}$ . This solves an open problem posed by Takhtajan-Teo in 2006 and investigated later by Figalli, Gay-Balmaz-Marsden-Ratiu and others. It turns out that there exists some quasisymmetric homeomorphism of the Weil-Petersson class which belongs neither to the Sobolev class  $H^{\frac{3}{2}}$  nor to the Lipschitz class  $\Lambda^1$ . Based on this new characterization of the Weil-Petersson class, a new metric is introduced on the Weil-Petersson Teichmüller space and is shown to be topologically equivalent to the Weil-Petersson metric.

## 1 INTRODUCTION

We begin with some basic definitions and notations. Let  $\Delta = \{z : |z| < 1\}$  denote the unit disk in the extended complex plane  $\hat{\mathbb{C}}$ .  $\Delta^* = \hat{\mathbb{C}} - \overline{\Delta}$  is the exterior of  $\Delta$ ,  $S^1 = \partial\Delta = \partial\Delta^*$  is the unit circle, and  $\mathbb{R}$  is the real line.

Let  $\text{Hom}^+(S^1)$  denote the set of all sense-preserving homeomorphisms of  $S^1$  onto itself. A homeomorphism  $h \in \text{Hom}^+(S^1)$  is said to be quasisymmetric if there exists some constant  $M > 0$  such that

$$(1.1) \quad \frac{1}{M} \leq \frac{|h(I_1)|}{|h(I_2)|} \leq M$$

for all pairs of adjacent arcs  $I_1$  and  $I_2$  on  $S^1$  with the same arc-length  $|I_1| = |I_2| (\leq \pi)$ . Beurling-Ahlfors [BA] proved that  $h \in \text{Hom}^+(S^1)$  is quasisymmetric if and only if there

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exists some quasiconformal homeomorphism of  $\Delta$  onto itself which has boundary values  $h$ . Later Douady-Earle [DE] gave a quasiconformal extension of  $h$  to the unit disk which is also conformally invariant.

The universal Teichmüller space  $T$  is a universal parameter space for all Riemann surfaces and can be defined as the space of all normalized quasisymmetric homeomorphisms on the unit circle, namely,  $T = \text{QS}(S^1)/\text{Möb}(S^1)$ . Here,  $\text{QS}(S^1)$  denotes the group of all quasisymmetric homeomorphisms of the unit circle, and  $\text{Möb}(S^1)$  the subgroup of Möbius transformations of the unit disk. It is known that the universal Teichmüller space plays a significant role in Teichmüller theory, and it is also a fundamental object in mathematics and in mathematical physics. On the other hand, several subclasses of quasisymmetric homeomorphisms and their Teichmüller spaces were introduced and studied for various purposes in the literature. We refer to the books [Ah], [FM], [Ga], [GL], [Hu], [IT], [Le], [Na], [Po2] and the papers [AZ], [Cu], [GS], [HS], [SW], [TT2] for an introduction to the subject and more details.

It is also well known that the universal Teichmüller space has a natural complex Banach manifold structure under which the hyperbolic Kobayashi metric is the classical Teichmüller metric (see [Ga], [Ro] or [EKK]). However, the Kobayashi-Teichmüller metric on any Teichmüller space is only induced from a Finsler structure (see [Ob]) and is not a Riemannian metric in general. On the other hand, there does exist a Riemannian metric on a finite dimensional Teichmüller space, the Weil-Petersson metric, which has attracted a good bit of attention (see [Hu], [IT], [TT2]). In order to extend the definition of the Weil-Petersson metric to the universal Teichmüller space, Nag-Verjovsky [NV] introduced a formal formula for the Weil-Petersson metric, which converges only at those vectors on the unit circle that belong to the Sobolev space  $H^{\frac{3}{2}}$ , however. To overcome this difficulty, Takhtajan-Teo [TT2] endowed the universal Teichmüller space with a new complex Hilbert manifold structure, under which the Weil-Petersson metric is a convergent Riemannian metric. But, under this new complex Hilbert manifold structure, the universal Teichmüller space  $T$  is not connected and has uncountably many connected components. In this paper, the component containing the identity map is called the Weil-Petersson Teichmüller space, which is denoted by  $T_0$ . Takhtajan-Teo [TT2] proved that, under the Weil-Petersson metric,  $T_0$  is precisely the completion of  $\text{Diff}_+(S^1)/\text{Möb}(S^1)$ , the space of all normalized  $C^\infty$  diffeomorphisms on the unit circle. It is known that the complex Fréchet manifold  $\text{Diff}_+(S^1)/\text{Möb}(S^1)$  plays an important role in one of the approaches to non-perturbative bosonic closed string field theory based on Kähler geometry (see [BR1-2]).

A quasi-symmetric homeomorphism which represents a point in  $T_0$  is said to belong to the Weil-Petersson class, which is denoted by  $\text{WP}(S^1)$ . Then  $T_0 = \text{WP}(S^1)/\text{Möb}(S^1)$ . Recall that a quasi-symmetric homeomorphism  $h$  belongs to  $\text{WP}(S^1)$  if and only if  $h$  has a quasiconformal extension  $f$  to the unit disk whose Beltrami coefficient  $\mu$  satisfies the property that  $\iint_\Delta |\mu(z)|^2 (1 - |z|^2)^{-2} dx dy < \infty$ . (see [Cu], [TT2]). The tangent space to  $T_0$  at the identity consists of precisely the  $H^{\frac{3}{2}}$  vector fields on the unit circle with some normalized conditions (see [NV], [TT2]). Due to their importance and wide applications to various areas such as mathematical physics and differential equation (see [GMR], [Ku],

[TT1]), the Weil-Petersson class and its Teichmüller space have been much investigated in recent years (see [Fi], [GMR], [HS], [Ku], [TT1-2], [Wu]). However, it is still an open problem how to characterize the regularity of the elements in  $WP(S^1)$ . This problem was proposed by Takhtajan-Teo in 2006 (see page 68 in [TT2]) and was investigated later by Figalli [Fi], Gay-Balmaz-Marsden-Ratiu [GMR] and some others. Since  $T_0$  is modeled on  $H^{\frac{3}{2}}$ , it is hoped that an element in  $WP(S^1)$  also has  $H^{\frac{3}{2}}$ -regularity (see [GMR]). In fact, based on the results by Figalli [Fi], Gay-Balmaz-Marsden-Ratiu [GMR] were able to prove that each homeomorphism in  $WP(S^1)$  belongs to  $H^{\frac{3}{2}-\epsilon}$  for each  $\epsilon > 0$ . However, we shall prove that the  $H^{\frac{3}{2}}$ -regularity may fail for a quasisymmetric homeomorphism in the Weil-Petersson class.

**Theorem 1.1.** *There exists some quasisymmetric homeomorphism in  $WP(S^1)$  which belongs neither to the Sobolev class  $H^{\frac{3}{2}}$  nor to the Lipschitz class  $\Lambda^1$ .*

The proof of Theorem 1.1 relies on the following result, which characterizes the regularity of a quasisymmetric homeomorphism in the Weil-Petersson class and solves the above-mentioned problem posed by Takhtajan-Teo [TT2].

**Theorem 1.2.** *A sense-preserving homeomorphism  $h$  on the unit circle belongs to the Weil-Petersson class  $WP(S^1)$  if and only if  $h$  is absolutely continuous (with respect to the arc-length measure) such that  $\log h'$  belongs to the Sobolev class  $H^{\frac{1}{2}}$ .*

It is easy to see that the Weil-Petersson class  $WP(S^1)$  can be generated by the flows of the  $H^{\frac{3}{2}}$  vector fields in the unit circle (see [GMR]). However, it is still an open problem whether or not the flows of the  $H^{\frac{3}{2}}$  vector fields are contained in  $WP(S^1)$ , though it is believed to be so (see [Fi]). Our Theorem 1.2 may shed some new light to this problem. In an attempt to study the regularity of the elements in  $WP(S^1)$ , Figalli [Fi] investigated the regularity of the flows of the  $H^{\frac{3}{2}}$  vector fields and showed that there exists some  $H^{\frac{3}{2}}$  vector field whose flow map is neither Lipschitz nor  $H^{\frac{3}{2}}$ . An immediate corollary of Theorem 1.1 is the following result, which improves Figalli's result in the sense that our flow map is also in  $WP(S^1)$ , while in Figalli's example, it is not known whether or not the flow map is in  $WP(S^1)$ .

**Corollary 1.3.** *There exists some  $H^{\frac{3}{2}}$  vector field whose flow map is in  $WP(S^1)$  but is neither Lipschitz nor  $H^{\frac{3}{2}}$ .*

Theorem 1.2 implies that a quasisymmetric homeomorphism  $h$  of the Weil-Petersson class is absolutely continuous such that  $\log h' \in H^{\frac{1}{2}}$ . This fact is hoped to be useful to the further study of the geometry and structure of  $T_0$ . As we shall see later (see Remark 5.6 below),  $WP(S^1)/S^1$  has a very simple model, namely, it can be identified as the real Hilbert space  $H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$  under the bijection  $h \mapsto \log |h'|$ . Based on this observation, we will introduce a new metric on  $T_0$ , which can be defined roughly as follows:

$$(1.2) \quad d(h_1, h_2) = \|\log |h'_1| - \log |h'_2|\|_{H^{\frac{1}{2}}}, \quad h_1, h_2 \in T_0.$$

The advantage of this metric is that, as a global metric, it gives directly the distances between two points in  $T_0$ . This is in contrast to the case for the Weil-Petersson metric,

which is an infinitesimal Riemann metric on the tangent bundle of  $T_0$ . Anyhow, we shall prove

**Theorem 1.4.** *The metric  $d$  and the Weil-Petersson metric induce the same topology on  $T_0$ .*

## 2 PRELIMINARIES ON THE WEIL-PETERSSON TEICHMÜLLER SPACE

In this section, we give some basic definitions and results on the Weil-Petersson Teichmüller space. The results turn out to be essential in the proof of Theorem 1.4. We follow the lines in our recent paper [SW], where the BMO theory of the universal Teichmüller space was investigated.

We begin with the standard theory of the universal Teichmüller space (see [Ah], [Le], [Na]). Let  $M(\Delta^*)$  denote the open unit ball of the Banach space  $L^\infty(\Delta^*)$  of essentially bounded measurable functions on  $\Delta^*$ . For  $\mu \in M(\Delta^*)$ , let  $f_\mu$  be the quasiconformal mapping on the extended plane  $\hat{\mathbb{C}}$  with complex dilatation equal to  $\mu$  in  $\Delta^*$ , conformal in  $\Delta$ , normalized by  $f_\mu(0) = f'_\mu(0) - 1 = f''_\mu(0) = 0$ . We say two elements  $\mu$  and  $\nu$  in  $M(\Delta^*)$  are equivalent, denoted by  $\mu \sim \nu$ , if  $f_\mu|_\Delta = f_\nu|_\Delta$ . Then  $T = M(\Delta^*)/\sim$  is the Bers model of the universal Teichmüller space. We let  $\Phi$  denote the natural projection from  $M(\Delta^*)$  onto  $T$  so that  $\Phi(\mu)$  is the equivalence class  $[\mu]$ .  $[0]$  is called the base point of  $T$ .

Let  $B_2(\Delta)$  denote the Banach space of functions  $\phi$  holomorphic in  $\Delta$  with norm

$$(2.1) \quad \|\phi\|_{B_2} = \sup_{z \in \Delta} (1 - |z|^2)^2 |\phi(z)|.$$

Consider the map  $S : M(\Delta^*) \rightarrow B_2(\Delta)$  which sends  $\mu$  to the Schwarzian derivative of  $f_\mu|_\Delta$ . Recall that for any locally univalent function  $f$ , its Schwarzian derivative  $S_f$  is defined by

$$(2.2) \quad S_f = N'_f - \frac{1}{2}N_f^2, \quad N_f = (\log f')'.$$

It is known that  $S$  is a holomorphic split submersion onto its image, which descends down to a map  $\beta : T \rightarrow B_2(\Delta)$  known as the Bers embedding. Via the Bers embedding,  $T$  carries a natural complex Banach manifold structure so that  $\Phi$  is a holomorphic split submersion.

We proceed to define the Weil-Petersson Teichmüller space (For details, see [TT2] and also [Cu]). We denote by  $\mathcal{L}(\Delta^*)$  the Banach space of all essentially bounded measurable functions  $\mu$  with norm

$$(2.3) \quad \|\mu\|_{\text{WP}} = \|\mu\|_\infty + \left( \frac{1}{\pi} \iint_{\Delta^*} \frac{|\mu(z)|^2}{(|z|^2 - 1)^2} dx dy \right)^{\frac{1}{2}}.$$

Set  $\mathcal{M}(\Delta^*) = M(\Delta^*) \cap \mathcal{L}(\Delta^*)$ . Then  $T_0 = \mathcal{M}(\Delta^*)/\sim$  is the Weil-Petersson Teichmüller space. Actually,  $T_0$  is the base point component of the universal Teichmüller space under the complex Hilbert manifold structure introduced by Takhtajan-Teo [TT2].

We denote by  $\mathcal{B}(\Delta)$  the Banach space of functions  $\phi$  holomorphic in  $\Delta$  with norm

$$(2.4) \quad \|\phi\|_{\mathcal{B}} = \left( \frac{1}{\pi} \iint_{\Delta} |\phi(z)|^2 (1 - |z|^2)^2 dx dy \right)^{\frac{1}{2}}.$$

Then,  $\mathcal{B}(\Delta) \subset B_2(\Delta)$ , and the inclusion map is continuous. Under the Bers projection  $S : M(\Delta^*) \rightarrow B_2(\Delta)$ ,  $S(\mathcal{M}(\Delta^*)) = S(M(\Delta^*)) \cap \mathcal{B}(\Delta)$ . Moreover, we have

**Proposition 2.1.**  *$S : \mathcal{M}(\Delta^*) \rightarrow \mathcal{B}(\Delta)$  is a holomorphic split submersion from  $\mathcal{M}(\Delta^*)$  onto its image. Consequently,  $T_0$  has a unique complex Hilbert manifold structure such that  $\beta : T_0 \rightarrow \mathcal{B}(\Delta)$  is a bi-holomorphic map from  $T_0$  onto a domain in  $\mathcal{B}(\Delta)$ . Under this complex Hilbert manifold structure, the natural projection  $\Phi$  from  $\mathcal{M}(\Delta^*)$  onto  $T_0$  is a holomorphic split submersion.*

It is well known that a quasiconformal self-mapping of  $\Delta^*$  induces a bi-holomorphic automorphism of the universal Teichmüller space (see [Le], [Na]). Precisely, let  $w : \Delta^* \rightarrow \Delta^*$  be a quasiconformal mapping with complex dilatation  $\mu$ . Then  $w$  induces an bi-holomorphic isomorphism  $R_w : M(\Delta^*) \rightarrow M(\Delta^*)$  as

$$(2.5) \quad R_w(\nu) = \left( \frac{\nu - \mu}{1 - \bar{\mu}\nu} \frac{\partial w}{\partial \bar{w}} \right) \circ w^{-1}.$$

$R_w$  descends down a bi-holomorphic isomorphism  $w^* : T \rightarrow T$  by  $w^* \circ \Phi = \Phi \circ R_w$ .

**Proposition 2.2.** *Suppose  $w$  is quasi-isometric under the Poincaré metric with Beltrami coefficient  $\mu \in \mathcal{M}(\Delta^*)$ . Then  $w^* : T_0 \rightarrow T_0$  is bi-holomorphic.*

*Proof.* Clearly,  $R_w$  maps  $\mathcal{M}(\Delta^*)$  into itself, and  $R_w : \mathcal{M}(\Delta^*) \rightarrow \mathcal{M}(\Delta^*)$  is bi-holomorphic. It follows from Proposition 2.1 that  $w^* : T_0 \rightarrow T_0$  is bi-holomorphic.  $\square$

We continue to consider the pre-logarithmic derivative model of the Weil-Petersson Teichmüller space. Let  $B(\Delta)$  denote the space of functions  $\phi$  holomorphic in  $\Delta$  with semi-norm

$$(2.6) \quad \|\phi\|_B = \sup_{z \in \Delta} (1 - |z|^2) |\phi'(z)|,$$

and  $B_0(\Delta)$  the subspace of  $B(\Delta)$  which consists of those  $\phi$  satisfying  $\lim_{|z| \rightarrow 1} (1 - |z|^2) \phi'(z) = 0$ . Recall that the pre-logarithmic derivative model  $\hat{T}$  of the universal Teichmüller space consists of all functions  $\log f'$  (in  $B(\Delta)$ ), where  $f$  belongs to the well known class  $S_Q$  of all univalent analytic functions  $f$  in the unit disk  $\Delta$  with the normalized condition  $f(0) = f'(0) - 1 = 0$  that can be extended to a quasiconformal mapping in the whole plane (see [AG], [Zhu]). Under the topology of Bloch norm (2.6),  $\hat{T}$  is a disconnected open set. Precisely,  $\hat{T} = \hat{T}_b \cup_{\theta \in [0, 2\pi)} \hat{T}_\theta$ , where  $\hat{T}_b = \{\log f' : f \in S_Q \text{ is bounded}\}$  and  $\hat{T}_\theta = \{\log f' : f \in S_Q \text{ satisfies } f(e^{i\theta}) = \infty\}$ ,  $\theta \in [0, 2\pi)$ , are the all connected components of  $\hat{T}$  (see [Zhu]). Each  $\hat{T}_\theta$  is a copy of the Bers model  $T$ , while  $\hat{T}_b$  is a fiber space over  $T$ . In fact,  $\hat{T}_b$  is a model of the universal Teichmüller curve (see [Ber], [Te]).

Let  $\mathcal{AD}(\Delta)$  denote the space of all functions  $\phi$  holomorphic in  $\Delta$  with semi-norm

$$(2.7) \quad \|\phi\|_{\mathcal{AD}} = \left( \frac{1}{\pi} \iint_{\Delta} |\phi'(z)|^2 dx dy \right)^{\frac{1}{2}},$$

and  $\mathcal{AD}_0(\Delta) = \{\phi \in \mathcal{AD}(\Delta) : \phi(0) = 0\}$ . Then  $\mathcal{AD}(\Delta) \subset B_0(\Delta)$ , and the inclusion map is continuous. We may define  $\mathcal{AD}(\Delta^*)$  similarly. Define

$$(2.8) \quad \Lambda(\phi) = \phi'' - \frac{1}{2}(\phi')^2, \phi \in \mathcal{AD}(\Delta).$$

Then it holds the following basic result.

**Lemma 2.3** ([TT2]).  $\Lambda : \mathcal{AD}(\Delta) \rightarrow \mathcal{B}(\Delta)$  is holomorphic.

We come back our situation. Fix  $z_0 \in \Delta^*$ . For  $\mu \in M(\Delta^*)$ , let  $g_\mu^{z_0}$  (abbreviated to be  $g_\mu$ ) be the quasiconformal mapping on the extended plane  $\hat{\mathbb{C}}$  with complex dilatation equal to  $\mu$  in  $\Delta^*$ , conformal in  $\Delta$ , normalized by  $g_\mu(0) = g'_\mu(0) - 1 = 0$ ,  $g_\mu(z_0) = \infty$ . Then  $\mu \sim \nu$  if and only if  $g_\mu|_{\Delta} = g_\nu|_{\Delta}$ . Consider the map  $L_{z_0}$  on  $M(\Delta^*)$  by setting  $L_{z_0}(\mu) = \log g'_\mu$ . Then  $\cup_{z_0 \in \Delta^*} L_{z_0}(\mathcal{M}(\Delta^*)) = \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ . We have the following result.

**Theorem 2.4.** For each  $z_0 \in \Delta^*$ ,  $L_{z_0} : \mathcal{M}(\Delta^*) \rightarrow \mathcal{AD}_0(\Delta)$  is holomorphic.

*Proof.* We first show  $L = L_{z_0} : \mathcal{M}(\Delta^*) \rightarrow \mathcal{AD}_0(\Delta)$  is continuous. Recall that  $L$  is continuous on  $M(\Delta^*)$  in the topology of Bloch norm (2.6)(see [Le]), namely,

$$(2.9) \quad \sup_{z \in \Delta} |N_{g_\nu}(z) - N_{g_\mu}(z)|(1 - |z|^2) \leq C(\|\mu\|_\infty)\|\nu - \mu\|_\infty, \mu, \nu \in M(\Delta^*).$$

Then,

$$\begin{aligned} \|L(\nu) - L(\mu)\|_{\mathcal{AD}}^2 &= \frac{1}{\pi} \iint_{\Delta} |N_{g_\nu}(z) - N_{g_\mu}(z)|^2 dx dy \\ &\leq C_1 \left( |N_{g_\nu}(0) - N_{g_\mu}(0)|^2 + \iint_{\Delta} (1 - |z|^2)^2 |N'_{g_\nu}(z) - N'_{g_\mu}(z)|^2 dx dy \right) \\ &\leq C_2 \left( \|\nu - \mu\|_\infty + \iint_{\Delta} (1 - |z|^2)^2 \left| (S_\nu(z) - S_\mu(z)) + \frac{1}{2}(N_{g_\nu}^2(z) - N_{g_\mu}^2(z)) \right|^2 dx dy \right) \\ &\leq C_3(\|\nu - \mu\|_\infty + \|S(\nu) - S(\mu)\|_{\mathcal{B}}^2 + (\|L(\nu)\|_{\mathcal{AD}}^2 + \|L(\mu)\|_{\mathcal{AD}}^2)\|\nu - \mu\|_\infty^2). \end{aligned}$$

By the holomorphy of  $S : \mathcal{M}(\Delta^*) \rightarrow \mathcal{B}(\Delta)$ , we conclude that  $L : \mathcal{M}(\Delta^*) \rightarrow \mathcal{AD}_0(\Delta)$  is continuous.

Now for each  $z \in \Delta$ , define  $l_z(\phi) = \phi(z)$  for  $\phi \in \mathcal{AD}_0(\Delta)$ . Then,  $l_z \in \mathcal{AD}_0^*(\Delta)$ , that is,  $l_z$  is a continuous linear functional on the Banach space  $\mathcal{AD}_0(\Delta)$ . Set  $A = \{l_z : z \in \Delta\}$ .  $A$  is a total subset of  $\mathcal{AD}_0^*(\Delta)$  in the sense that  $l_z(\phi) = 0$  for all  $z \in \Delta$  implies that  $\phi = 0$ . Now for each  $z \in \Delta$ , each pair  $(\mu, \nu) \in \mathcal{M}(\Delta^*) \times \mathcal{L}(\Delta^*)$  and small  $t$  in the complex plane, by the well known holomorphic dependence of quasiconformal mappings on parameters (see [Ah], [Le], [Na]), we conclude that  $l_z(L(\mu + t\nu)) = L(\mu + t\nu)(z)$  is a holomorphic function of  $t$ . By a general result about the infinite dimensional holomorphy (see [Le], [Na]), it follows that  $L : \mathcal{M}(\Delta^*) \rightarrow \mathcal{AD}_0(\Delta)$  is holomorphic.  $\square$

**Theorem 2.5.**  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$  is a connected open subset of  $\mathcal{AD}_0(\Delta)$ , and  $\Lambda$  is a holomorphic split submersion from  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$  onto  $\beta(T_0)$ .

*Proof.* Clearly,  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$  is an open subset of  $\mathcal{AD}_0(\Delta)$ . We need to show that each point of  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$  can be connected to 0 by a path in  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ .

Let  $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ . Then  $f$  can be extended to a quasiconformal mapping in the whole plane whose Beltrami coefficient  $\mu$  belongs to  $\mathcal{M}(\Delta^*)$ , and  $z_0 = f^{-1}(\infty) \in \Delta^*$ . For each  $t \in [0, 1]$ , let  $f_t \in S_Q$  be the unique mapping whose quasiconformal extension to the whole plane has Beltrami coefficient  $t\mu$ , and  $f_t(z_0) = \infty$ . Theorem 2.4 implies that  $\log f'_t$ ,  $t \in [0, 1]$ , is a path in  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$  joining  $\log f'_0$  to  $\log f'$ . Now, if  $z_0 = \infty$ , then  $f_0(z) = z$ , and we are done. If  $z_0 \neq \infty$ , then  $f_0(z) = z_0 z / (z_0 - z)$ , and  $\log f'_0(r \cdot)$ ,  $r \in [0, 1]$ , is a curve in  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$  connecting 0 and  $\log f'_0$ .

Clearly, Lemma 2.3 implies that  $\Lambda$  is holomorphic on  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ . Choose  $z_0 \in \Delta^*$ . Since  $S = \Lambda \circ L_{z_0}$ , we conclude that  $\Lambda$  is a holomorphic split submersion from  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$  onto  $\beta(T_0)$  since  $L_{z_0} : \mathcal{M}(\Delta^*) \rightarrow \hat{T}_b \cap \mathcal{AD}_0(\Delta)$  is holomorphic, and  $S : \mathcal{M}(\Delta^*) \rightarrow \beta(T_0)$  is a holomorphic split submersion.  $\square$

### 3 SOME LEMMAS

In this section, we give some lemmas needed to prove Theorems 1.1 and 1.2. First we recall some basic definitions and results on Sobolev spaces, the harmonic Dirichlet space and the BMO space that will be frequently used in the rest of the paper (see [Gar], [RS], [Tr]).

For any  $s > 0$ , the Sobolev space  $H^s$  consists of all integrable functions  $u \in L^1([0, 2\pi])$  on the unit circle with semi-norm

$$(3.1) \quad \|u\|_{H^s} = \left( \sum_{n=-\infty}^{+\infty} |n|^{2s} |a_n(u)|^2 \right)^{\frac{1}{2}},$$

where, as usual,  $a_n(u)$  is the  $n$ -th Fourier coefficient of  $u$ , namely,

$$(3.2) \quad a_n(u) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) e^{-in\theta} d\theta.$$

In this paper, the two cases we are concerned are  $s = \frac{3}{2}$  and  $s = \frac{1}{2}$ . Recall that  $u \in H^{\frac{3}{2}}$  if and only if  $u$  is absolutely continuous with  $u' \in H^{\frac{1}{2}}$ . It is also known that an integrable function  $u$  on the unit circle belongs to  $H^{\frac{1}{2}}$  if and only if

$$(3.3) \quad \int_0^{2\pi} \int_0^{2\pi} \frac{|u(s) - u(t)|^2}{|\sin((s-t)/2)|^2} ds dt < +\infty.$$

We need another description of the space  $H^{\frac{1}{2}}$ . Let  $\mathcal{D}(\Delta)$  denote the space of all harmonic functions  $u$  in the unit disk  $\Delta$  with semi-norm

$$(3.4) \quad \|u\|_{\mathcal{D}} = \left( \frac{1}{\pi} \iint_{\Delta} (|\partial_z u|^2 + |\partial_{\bar{z}} u|^2) dx dy \right)^{\frac{1}{2}}.$$

Then,  $\mathcal{D}(\Delta) = \mathcal{AD}(\Delta) \oplus \overline{\mathcal{AD}(\Delta)}$ , or precisely, for each  $u \in \mathcal{D}(\Delta)$ , there exists a unique pair of holomorphic functions  $\phi$  and  $\psi$  in  $\mathcal{AD}(\Delta)$  with  $\phi(0) - u(0) = \psi(0) = 0$  such that  $u = \phi + \bar{\psi}$ . Here it is a convenient place to introduce two basic operators on the Dirichlet space  $\mathcal{D}(\Delta)$ . They are  $P^+$  and  $P^-$ , defined respectively by  $P^+u = \phi$  and  $P^-u = \bar{\psi}(\bar{z})$  for  $u = \phi + \bar{\psi}$ . It is well known that each function  $u \in \mathcal{D}(\Delta)$  has boundary values almost everywhere on the unit circle, and the boundary function  $u(e^{i\theta})$  belongs to  $H^{\frac{1}{2}}$ , and conversely each function in  $H^{\frac{1}{2}}$  is obtained in this way (see [Zy]). In fact, the usual Poisson integral operator  $P$  establishes a one-to-one map from  $H^{\frac{1}{2}}$  onto  $\mathcal{D}(\Delta)$  with  $\|Pu\|_{\mathcal{D}} = \|u\|_{H^{\frac{1}{2}}}$ .

Let  $I_0$  be a finite interval on the real line. An integrable function  $u \in L^1(I_0)$  is said to have bounded mean oscillation if

$$(3.5) \quad \|u\|_{\text{BMO}} = \sup \frac{1}{|I|} \int_I |u(t) - u_I| dt < +\infty,$$

where the supremum is taken over all sub-intervals  $I$  of  $I_0$ , while  $u_I$  is the average of  $u$  on the interval  $I$ , namely,

$$(3.6) \quad u_I = \frac{1}{|I|} \int_I u(t) dt.$$

If  $u$  also satisfies the condition

$$(3.7) \quad \lim_{|I| \rightarrow 0} \frac{1}{|I|} \int_I |u(t) - u_I| dt = 0,$$

we say  $u$  has vanishing mean oscillation. These functions are denoted by BMO and VMO, respectively. In the following, unless specified, these functions are considered on the unit circle, namely,  $I_0 = [0, 2\pi]$ . Then  $H^{\frac{1}{2}} \subset \text{VMO}$ , and the inclusion map is continuous (see [Zh]).

We need some basic results on BMO functions. By the well-known theorem of John-Nirenberg for BMO functions (see [Gar]), there exist two universal positive constants  $C_1$  and  $C_2$  such that for any BMO function  $u$ , any subinterval  $I$  of  $I_0$  and any  $\lambda > 0$ , it holds that

$$(3.8) \quad \frac{|\{t \in I : |u(t) - u_I| \geq \lambda\}|}{|I|} \leq C_1 \exp\left(\frac{-C_2\lambda}{\|u\|_{\text{BMO}}}\right).$$

For any  $p \geq 1$ , by Chebychev's inequality, we have

$$\begin{aligned} \frac{1}{|I|} \int_I (e^{|u - u_I|} - 1)^p dt &= \frac{1}{|I|} \int_0^\infty |\{t \in I : |u - u_I| \geq \lambda\}| d((e^\lambda - 1)^p) \\ &\leq pC_1 \int_0^\infty (e^\lambda - 1)^{p-1} e^\lambda \exp\left(\frac{-C_2\lambda}{\|u\|_{\text{BMO}}}\right) d\lambda. \end{aligned}$$

When  $p\|u\|_{\text{BMO}} < C_2$ , we obtain

$$(3.9) \quad \frac{1}{|I|} \int_I (e^{|u - u_I|} - 1)^p dt \leq \frac{pC_1\|u\|_{\text{BMO}}}{C_2 - p\|u\|_{\text{BMO}}}.$$

We will repeatedly use the following basic result:



**Lemma 3.1.** *Let  $u \in \text{BMO}$  and  $p \geq 1$ . Then  $e^u \in L^p(I_0)$  when  $p\|u\|_{\text{BMO}}$  is small. In particular, if  $u \in \text{VMO}$ , then  $e^u \in L^p(I_0)$  for any real number  $p \geq 1$ .*

*Proof.* When  $p\|u\|_{\text{BMO}} < C_2$ , it follows from (3.9) that

$$(3.10) \quad \|e^{u-u_{I_0}} - 1\|_p^p = \frac{1}{|I_0|} \int_{I_0} |e^{u-u_{I_0}} - 1|^p dt \leq \frac{pC_1\|u\|_{\text{BMO}}}{C_2 - p\|u\|_{\text{BMO}}}.$$

Consequently,

$$\|e^u\|_p \leq e^{\|u\|_1} (\|e^{u-u_{I_0}} - 1\|_p + |I_0|^{\frac{1}{p}}) < +\infty.$$

Now suppose  $u \in \text{VMO}$ , and  $p \geq 1$  is any real number. By (3.7), for any sufficiently small subinterval  $I$  of  $I_0$ ,  $u$  has small BMO norm on  $I$  so that  $e^u \in L^p(I)$ . Decompose  $I_0$  as the union of finitely many small subintervals  $I_j$  so that  $e^u \in L^p(I_j)$ , we conclude that  $e^u \in L^p(I_0)$  as required.  $\square$

**Lemma 3.2.** *Let  $u \in \text{VMO}$  and  $u_n \in \text{BMO}$  on the unit circle. If  $\|u_n - u\|_{\text{BMO}} \rightarrow 0$  and  $a_0(u_n - u) \rightarrow 0$  when  $n \rightarrow \infty$ , then for any  $p \geq 1$ ,  $\|e^{u_n} - e^u\|_p \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* By (3.10),

$$\left\| e^{(u_n-u)-a_0(u_n-u)} - 1 \right\|_{2p}^{2p} \leq \frac{2pC_1\|u_n - u\|_{\text{BMO}}}{C_2 - 2p\|u_n - u\|_{\text{BMO}}} \rightarrow 0, \quad n \rightarrow \infty.$$

On the other hand, since  $v \in \text{VMO}$ , Lemma 3.1 implies that  $e^u \in L^{2p}([0, 2\pi])$ . Consequently,

$$\begin{aligned} \|e^{u_n} - e^u\|_p &\leq \|e^{u_n-u} - 1\|_{2p} \|e^u\|_{2p} \\ &\leq \|e^u\|_{2p} \left( e^{a_0(u_n-u)} \|e^{(u_n-u)-a_0(u_n-u)} - 1\|_{2p} + \|e^{a_0(u_n-u)} - 1\|_{2p} \right), \end{aligned}$$

which implies  $\|e^{u_n} - e^u\|_p \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

Recall that for each sense-preserving homeomorphisms  $h$  of the unit circle onto itself, there exists some strictly increasing continuous function  $\phi$  on the real line with  $\phi(\theta + 2\pi) - \phi(\theta) \equiv 2\pi$  such that  $h(e^{i\theta}) = e^{i\phi(\theta)}$ . When  $\phi$  is differentiable at  $\theta$ , we say that  $h$  is differentiable at  $e^{i\theta}$ , and set

$$(3.11) \quad h'(e^{i\theta}) = e^{i(\phi(\theta)-\theta)} \phi'(\theta).$$

We say  $h$  is absolutely continuous on the unit circle if  $\phi$  is locally absolutely continuous on the real line.

**Lemma 3.3.** *Let  $h$  be an absolutely continuous sense-preserving homeomorphism on the unit circle such that  $\log h' \in \text{VMO}$ . Then  $h$  is a quasymmetric homeomorphism.*

*Proof.* Partyka (see [Pa1], Theorem 3.4.7) asserted that  $h$  is actually a symmetric homeomorphism in the sense of Gardiner-Sullivan [GS], namely, for any pair of adjacent subintervals  $I_1$  and  $I_2$  in  $[0, 2\pi]$  with  $|I_1| = |I_2|$ , it holds that

$$(3.12) \quad \frac{|h(I_1)|}{|h(I_2)|} = 1 + o(1), \quad |I_1| = |I_2| \rightarrow 0 + .$$

A detailed proof of this fact was given in [Pa2]. Here we give a fast proof for completeness.

Let  $h(e^{i\theta}) = e^{i\phi(\theta)}$  as above, and set  $v = \log \phi' = \log |h'|$  for simplicity. Then  $v \in \text{VMO}$ . For any small subinterval  $I$  in  $[0, 2\pi]$  such that the BMO-norm of  $v$  on  $I$  is small, we conclude by (3.9) (with  $p = 1$ ) that

$$(3.13) \quad \int_I e^{|v-v_I|} dt \leq |I| \left( 1 + \frac{C_1 \|v|_I\|_{\text{BMO}}}{C_2 - \|v|_I\|_{\text{BMO}}} \right) = |I|(1 + o(1)), \quad |I| \rightarrow 0.$$

Noting that

$$|h(I)| = \int_I \phi'(t) dt = \int_I e^v dt = e^{v_I} \int_I e^{v-v_I} dt,$$

we obtain from (3.13) that, as  $|I| \rightarrow 0$ ,

$$|h(I)| \leq e^{v_I} \int_I e^{|v-v_I|} dt \leq |I| e^{v_I} (1 + o(1)),$$

$$|h(I)| \geq e^{v_I} \int_I e^{-|v-v_I|} dt \geq \frac{|I|^2 e^{v_I}}{\int_I e^{|v-v_I|} dt} \geq |I| e^{v_I} (1 + o(1)),$$

and so

$$(3.14) \quad |h(I)| = |I| e^{v_I} (1 + o(1)), \quad |I| \rightarrow 0.$$

Now let  $I_1$  and  $I_2$  be two adjacent subintervals in  $[0, 2\pi]$  with  $|I_1| = |I_2| = l$  being small such that the BMO-norm of  $v$  on  $I_1 \cup I_2$  is small. It holds that (see [Gar], (1.3) in Chapter VI)

$$(3.15) \quad |v_{I_1} - v_{I_2}| = 2|v_{I_1} - v_{I_1 \cup I_2}| \leq 4\|v|_{I_1 \cup I_2}\|_{\text{BMO}} = o(1), \quad l \rightarrow 0 + .$$

Then (3.12) follows from (3.14-3.15) immediately.  $\square$

**Lemma 3.4.** *Let  $h$  be an absolutely continuous sense-preserving homeomorphism on the unit circle. Then  $\log h' \in H^{\frac{1}{2}}$  if and only if  $\log |h'| \in H^{\frac{1}{2}}$ .*

*Proof.* Let  $h(e^{i\theta}) = e^{i\phi(\theta)}$  as before. Without loss of generality, we assume that  $h(1) = 1$  so that  $\phi(0) = 0$ ,  $\phi(2\pi) = 2\pi$ . Then  $|h'(e^{i\theta})| = \phi'(\theta)$ , and

$$(3.16) \quad \log h'(e^{i\theta}) = \log |h'(e^{i\theta})| + i(\phi(\theta) - \theta).$$

It is clear that  $\log |h'| \in H^{\frac{1}{2}}$  if  $\log h' \in H^{\frac{1}{2}}$ .

Conversely, we suppose that  $\log \phi' = \log |h'| \in H^{\frac{1}{2}}$ . We shall show that  $\phi(\theta) - \theta \in H^1$ , which implies that  $\log h' \in H^{\frac{1}{2}}$ . In fact, the  $n$ -th ( $n \neq 0$ ) Fourier coefficient of  $\phi(\theta) - \theta$  is

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} (\phi(\theta) - \theta) e^{-in\theta} d\theta = \frac{1}{2n\pi i} \int_0^{2\pi} (\phi'(\theta) - 1) e^{-in\theta} d\theta.$$

Thus, by Parseval's equality, we conclude by Lemma 3.1 that

$$\sum_{n \neq 0} n^2 |a_n|^2 = \frac{1}{4\pi^2} \sum_{n \neq 0} \left| \int_0^{2\pi} (\phi'(\theta) - 1) e^{-in\theta} d\theta \right|^2 = \|\phi' - 1\|_2^2 < +\infty.$$

This completes the proof.  $\square$

#### 4 PROOF OF THEOREM 1.1

In this section, we will prove Theorem 1.1 by means of Theorem 1.2, whose proof will be given in the next section. Combining with Theorem 1.2, the following result gives the proof of Theorem 1.1.

**Theorem 4.1.** *Define  $h(e^{i\theta}) = e^{i\varphi(\theta)}$  by*

$$(4.1) \quad \varphi(\theta) = c \int_0^\theta \left( \log^2 \sin \frac{t}{2} + \frac{(\pi - t)^2}{4} \right) dt, \quad \theta \in [0, 2\pi],$$

where  $c > 0$  is a constant so that  $\varphi(2\pi) = 2\pi$ . Then  $h$  is a sense-preserving homeomorphism which is absolutely continuous such that  $\log h' \in H^{\frac{1}{2}}$ , but  $h$  is neither  $H^{\frac{3}{2}}$  nor Lipschitz.

*Proof.* We consider

$$(4.2) \quad g(z) = \log \log \frac{2}{1-z}.$$

$g$  is holomorphic in  $\Delta$ , and except for  $e^{i\theta} = 1$ ,  $\lim_{z \rightarrow e^{i\theta}} g(z)$  exists and equals

$$g(e^{i\theta}) = \log \left( -\log \sin \frac{\theta}{2} + i \frac{\pi - \theta}{2} \right).$$

We first show that  $g \in \mathcal{AD}(\Delta)$ . Noting that

$$g'(z) = \frac{1}{(1-z) \log \frac{2}{1-z}},$$

it is sufficient to show that

$$\iint_{\{|z-1|<1, \Re z < 1\}} |g'(z)|^2 < +\infty.$$

This can be done as follows:

$$\begin{aligned}
\iint_{\{|z-1|<1, \Re z<1\}} |g'(z)|^2 &= \int_{\{|w|<1, \Re w>0\}} \frac{1}{|w \log \frac{2}{w}|^2} \\
&= 2 \int_0^1 \rho d\rho \int_0^{\frac{\pi}{2}} \frac{d\theta}{\rho^2 (\log^2 \frac{2}{\rho} + \theta^2)} \\
&= 2 \int_0^1 \frac{1}{\rho \log \frac{2}{\rho}} \arctan \frac{\pi}{2 \log^2 \frac{2}{\rho}} \\
&= 2 \int_{\log 2}^{+\infty} \frac{\arctan \frac{\pi}{2x}}{x} dx \\
&< \pi \int_{\log 2}^{+\infty} \frac{1}{x^2} dx = \frac{\pi}{\log 2}.
\end{aligned}$$

Thus,  $g \in H^{\frac{1}{2}}$ , which implies that  $\Re g \in H^{\frac{1}{2}}$ . By Lemma 3.1, we obtain that  $\exp(2\Re g) \in L^1([0, 2\pi])$ . Noting that

$$\Re g(e^{i\theta}) = \log \left| -\log \sin \frac{\theta}{2} + i \frac{\pi - \theta}{2} \right| = \frac{1}{2} \log \left( \log^2 \sin \frac{\theta}{2} + \frac{(\pi - \theta)^2}{4} \right),$$

we conclude that our function  $\varphi$  defined in (4.1) is well-defined, strictly increasing and absolutely continuous with

$$(4.3) \quad \varphi'(\theta) = c \left( \log^2 \sin \frac{\theta}{2} + \frac{(\pi - \theta)^2}{4} \right).$$

Thus,  $h$  is an absolutely continuous sense-preserving homeomorphism of the unit circle onto itself. Since  $\|\varphi'\|_\infty = \infty$ ,  $h$  is not Lipschitz. On the other hand, since  $\log \varphi' = \log c + 2\Re g \in H^{\frac{1}{2}}$ , we conclude by Lemma 3.4 that  $\log h' \in H^{\frac{1}{2}}$ .

It remains to show that  $h$  is not in  $H^{\frac{3}{2}}$ . By means of (3.1) and (3.2), it is sufficient to show that  $\varphi'$  is not in  $H^{\frac{1}{2}}$ . In fact, by (3.1), for each homeomorphism  $h(e^{i\theta}) = e^{i\phi(\theta)}$  of the unit circle,  $h(e^{i\theta}) \in H^{\frac{3}{2}}$  if and only if  $(h(e^{i\theta}))' = e^{i\phi(\theta)} \phi'(\theta) \in H^{\frac{1}{2}}$ . On the other hand, noting that

$$|e^{i\phi(s)} \phi'(s) - e^{i\phi(t)} \phi'(t)| \geq |\phi'(s) - \phi'(t)|,$$

we conclude by (3.2) that  $e^{i\phi(\theta)} \phi'(\theta) \in H^{\frac{1}{2}}$  implies  $\phi'(\theta) \in H^{\frac{1}{2}}$ .

To prove  $\varphi'$  is not in  $H^{\frac{1}{2}}$ , we consider the following analytic function in the unit disk

$$(4.4) \quad f(z) = \log(1 - z).$$

Then, except for  $e^{i\theta} = 1$ ,  $\lim_{z \rightarrow e^{i\theta}} f(z)$  exists and is equal to

$$(4.5) \quad f(e^{i\theta}) = \log(1 - e^{i\theta}) = \log 2 + \log \sin \frac{\theta}{2} - i \frac{\pi - \theta}{2}.$$

It is easy to see that  $f$  does not belong to  $\mathcal{AD}(\Delta)$ , which implies that  $\Re f$  is not in  $H^{\frac{1}{2}}$ . Thus,  $\log \sin(\theta/2)$  is not in  $H^{\frac{1}{2}}$ . By (3.3) we have

$$(4.6) \quad \int_0^\pi \int_0^\pi \frac{|\log \sin s - \log \sin t|^2}{|\sin(s-t)|^2} ds dt = +\infty.$$

Fix  $0 < \epsilon < \pi/4$ , and set  $I_\epsilon = [\pi/2 - \epsilon, \pi/2 + \epsilon]$ ,  $(I_\epsilon \times I_\epsilon)^c = [0, \pi] \times [0, \pi] - I_\epsilon \times I_\epsilon$ . Noting that  $\log(1+x) < x$  when  $x > 0$ , we find that

$$|\log x - \log y| \leq \frac{|x - y|}{\min(x, y)}, \quad x > 0, y > 0.$$

On the other hand,  $\sin x \geq 2/\pi x$  when  $0 < x < \pi/2$ , we conclude that

$$\frac{|\log \sin s - \log \sin t|^2}{|\sin(s-t)|^2} \leq \frac{\pi^2}{4} \frac{|\sin s - \sin t|^2}{|s-t|^2 \min(\sin^2 s, \sin^2 t)} \leq \frac{\pi^2}{4 \cos^2 \epsilon} \leq \frac{\pi^2}{2}, \quad s, t \in I_\epsilon.$$

Thus,

$$\int_{I_\epsilon} \int_{I_\epsilon} \frac{|\log \sin s - \log \sin t|^2}{|\sin(s-t)|^2} ds dt < +\infty.$$

It follows from (4.6) that

$$\iint_{(I_\epsilon \times I_\epsilon)^c} \frac{|\log \sin s - \log \sin t|^2}{|\sin(s-t)|^2} ds dt = +\infty.$$

Noting that  $\log \sin s < \log \cos \epsilon < 0$  when  $s \in I_\epsilon^c$ , we conclude from the above equality that

$$\begin{aligned} & \iint_{(I_\epsilon \times I_\epsilon)^c} \frac{|\log^2 \sin s - \log^2 \sin t|^2}{|\sin(s-t)|^2} ds dt \\ & \geq \log^2 \cos \epsilon \iint_{(I_\epsilon \times I_\epsilon)^c} \frac{|\log \sin s - \log \sin t|^2}{|\sin(s-t)|^2} ds dt = +\infty. \end{aligned}$$

By (3.3), this implies that  $\log^2 \sin(\theta/2)$  is not in  $H^{\frac{1}{2}}$ .

Now it follows easily from (3.1) that the second part of  $\varphi'$  in (4.3) is in  $H^{\frac{1}{2}}$ . Actually, a direct computation will show that the  $n$ -th ( $n \neq 0$ ) Fourier coefficient of  $(\pi - \theta)^2$  is

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} (\pi - \theta)^2 e^{-in\theta} d\theta = \frac{2}{n^2}.$$

Consequently,  $\varphi'$  is not in  $H^{\frac{1}{2}}$ . This completes the proof of Theorem 4.1.  $\square$

Now a natural question is whether an  $H^{\frac{3}{2}}$  quasymmetric homeomorphism  $h$  satisfies  $\log h' \in H^{\frac{1}{2}}$  so that it belongs to the Weil-Petersson class. The following result gives a stronger answer in the negative. Recall that an infinitely differentiable function on the unit circle belongs to  $H^s$  for any  $s > 0$ .

**Proposition 4.2.** *There exists an infinitely differentiable symmetric homeomorphism  $h$  such that  $\log h'$  does not belong to VMO.*

*Proof.* Define  $h(e^{i\theta}) = e^{i\phi(\theta)}$  by

$$\phi(\theta) = \theta - \sin \theta.$$

Clearly,  $h$  is an infinitely differentiable homeomorphism of the unit circle, and  $\phi'(\theta) = 1 - \cos \theta = 2 \sin^2(\theta/2)$ . Consider the analytic function  $f$  we defined by (4.4) during the proof of Theorem 4.1. It is easy to see that  $f$  even does not belong to  $B_0(\Delta)$ , which implies that  $\Re f$  is not in VMO (see [Gar], [Po2], [Zhu]). So  $\log \sin(\theta/2)$  is not in VMO. Thus,  $\log \phi'$  and consequently  $\log h'$  is not in VMO. It remains to show that  $h$  is symmetric.

For  $\theta \in [0, 2\pi]$  and small  $t$ ,

$$\phi(\theta + t) - \phi(\theta) = t - \sin(\theta + t) + \sin \theta = (1 - \cos \theta)t + o(1)t,$$

where  $o(1)$  tends to zero uniformly for  $\theta$  as  $t \rightarrow 0$ . Then, for any pair of adjacent arcs  $I_1$  and  $I_2$  on  $S^1$  with small arc-length  $|I_1| = |I_2|$ ,

$$\frac{|h(I_1)|}{|h(I_2)|} = 1 + o(1), \quad |I_1| = |I_2| \rightarrow 0.$$

So  $h$  is a symmetric homeomorphism.  $\square$

## 5 PROOF OF THEOREM 1.2

In this section, we will give the proof of Theorem 1.2. The proof is based on the study of the pull-back operator on the Sobolev space  $H^{\frac{1}{2}}$  and also on the Dirichlet space  $\mathcal{D}(\Delta)$  by a quasisymmetric homeomorphism.

Let  $h$  be a quasisymmetric homeomorphism. Then  $h$  induces a pull-back operator by

$$(5.1) \quad P_h u = u \circ h, \quad u \in H^{\frac{1}{2}}.$$

$P_h$  is a bounded isomorphism from  $H^{\frac{1}{2}}$  onto itself, and  $P_h^{-1} = P_{h^{-1}}$ . This operator has played an important role in the study of Teichmüller theory (see [HS], [NS], [Pa1], [SW], [TT2]). In particular, Nag-Sullivan [NS] proved that the universal Teichmüller space  $T$  can be embedded in the universal Siegel period matrix space by means of the operator  $P_h$  (see also [TT2]). Notice that  $P_h$  (or more precisely,  $P \circ P_h$ , the composition of  $P_h$  with the Poisson integral operator  $P$ ) is also a bounded isomorphism from  $\mathcal{D}(\Delta)$  onto itself, and  $P_h^{-1} = P_{h^{-1}}$ .

**Proof of “only if” part:** Recall that for any quasisymmetric homeomorphism  $h$ , there exists a unique pair of conformal mappings  $f \in S_Q$  and  $g$  on  $\Delta$  and  $\Delta^*$ , respectively, such that  $f(0) = f'(0) - 1 = 0$ ,  $g(\infty) = \infty$ ,  $h = f^{-1} \circ g$  on  $S^1$ . We call this a normalized

decomposition of  $h$ . Conversely, for each  $f \in S_Q$ , there exists a quasisymmetric  $h$  with the normalized decomposition  $h = f^{-1} \circ g$ . It is clear that  $h$  is uniquely determined if  $h(1) = 1$ , and in this case we say  $h$  is the normalized conformal sewing mapping of  $f$ .

Now suppose  $h \in \text{WP}(S^1)$ . Consider the above normalized decomposition  $h = f^{-1} \circ g$ . Then,  $\log f' \in \mathcal{AD}(\Delta)$ ,  $\log g' \in \mathcal{AD}(\Delta^*)$ . For details, see [TT2] and also [Cu]. Then,  $h$  is absolutely continuous on  $S^1$ , and from  $f \circ h = g$  we obtain  $(f' \circ h)h' = g'$ . Thus,

$$(5.2) \quad \log h' = \log g' - \log f' \circ h = \log g' - P_h \log f'.$$

Consequently,  $\log h' \in H^{\frac{1}{2}}$ .  $\square$

The proof of the another direction is more difficult. We need to investigate further the pull back operator  $P_h$  induced by a quasisymmetric homeomorphism. When restricted to  $\mathcal{AD}(\Delta)$ ,  $P_h$  (more precisely,  $P \circ P_h$ ) is a bounded operator from  $\mathcal{AD}(\Delta)$  into  $\mathcal{D}(\Delta)$ . So we may define two further operators  $P_h^+ = P^+ \circ P_h$  and  $P_h^- = P^- \circ P_h$ . Both  $P_h^+$  and  $P_h^-$  are bounded operators from  $\mathcal{AD}(\Delta)$  into itself. For completeness, we recall that  $P_h^-$  is a compact operator if and only if  $h$  is symmetric, while  $P_h^-$  is a Hilbert-Schmidt operator if and only if  $h$  belongs to the Weil-Petersson class  $\text{WP}(S^1)$  (see [HS]). We will not use this result in this paper.

The following result will play an important role in our proof.

**Proposition 5.1.**  *$P_h^+$  is a bounded isomorphism from  $\mathcal{AD}(\Delta)$  onto itself. Moreover, it holds that*

$$(5.3) \quad \|P_h^+ \phi\|_{\mathcal{AD}}^2 = \|\phi\|_{\mathcal{AD}}^2 + \|P_h^- \phi\|_{\mathcal{AD}}^2, \quad \phi \in \mathcal{AD}(\Delta).$$

*Proof.* The proof is based on the observations in our previous papers [HS] and [SW]. For completeness, we give the details here.

Let  $\mathcal{A}^2(\Delta)$  denote the complex Hilbert space of all holomorphic functions  $\psi$  on the unit disk with norm

$$(5.4) \quad \|\psi\|_{\mathcal{A}^2} = \left( \frac{1}{\pi} \iint_{\Delta} |\psi(\zeta)|^2 d\xi d\eta \right)^{\frac{1}{2}}.$$

Then,  $D\phi(z) = \phi'(z)$  defines an isometric isomorphism from  $\mathcal{AD}_0(\Delta)$  onto  $\mathcal{A}^2(\Delta)$ .

For a quasisymmetric homeomorphism  $h$ , two kernel functions were introduced in the previous paper [HS] by Hu and the author. They are

$$(5.5) \quad \phi_h(\zeta, z) = \frac{1}{2\pi i} \int_{S^1} \frac{h(w)}{(1 - \zeta w)^2 (1 - zh(w))} dw, \quad (\zeta, z) \in \Delta \times \Delta,$$

$$(5.6) \quad \psi_h(\zeta, z) = \frac{1}{2\pi i} \int_{S^1} \frac{h(w)}{(\zeta - w)^2 (1 - zh(w))} dw, \quad (\zeta, z) \in \Delta \times \Delta.$$

These two kernels induce two bounded operators on  $\mathcal{A}^2(\Delta)$  as follows:

$$(5.7) \quad T_h^- \psi(\zeta) = \frac{1}{\pi} \iint_{\Delta} \phi_h(\zeta, \bar{z}) \psi(z) dx dy, \quad \psi \in \mathcal{A}^2(\Delta), \zeta \in \Delta,$$

and

$$(5.8) \quad T_h^+ \psi(\zeta) = \frac{1}{\pi} \iint_{\Delta} \psi_h(\zeta, \bar{z}) \psi(z) dx dy, \quad \psi \in \mathcal{A}^2(\Delta), \zeta \in \Delta.$$

Then, Theorem 3.1 in [HS] says that on  $\mathcal{AD}(\Delta)$ ,

$$(5.9) \quad D \circ P_h^- = T_h^- \circ D, \quad D \circ P_h^+ = T_h^+ \circ D,$$

while Lemma 2.3 in [SW] says that

$$(5.10) \quad \|T_h^+ \psi\|_{\mathcal{A}^2}^2 = \|\psi\|_{\mathcal{A}^2}^2 + \|T_h^- \psi\|_{\mathcal{A}^2}^2, \quad \psi \in \mathcal{A}^2(\Delta).$$

(5.3) follows from (5.9) and (5.10) immediately. It remains to show that  $P_h^+$  is surjective.

Consider the decomposition  $h = f^{-1} \circ g$  as above. Set

$$(5.11) \quad U(f, \zeta, z) = \frac{f'(\zeta)f'(z)}{[f(\zeta) - f(z)]^2} - \frac{1}{(\zeta - z)^2}, \quad (\zeta, z) \in \Delta \times \Delta.$$

Then  $S_f(z) = -6U(f, z, z)$  is the Schwarzian derivative of  $f$ .  $f$  determines the so-called Grunsky operator on  $\mathcal{A}^2(\Delta)$ , defined as

$$(5.12) \quad G_f \psi(\zeta) = \frac{1}{\pi} \iint_{\Delta} U(f, \zeta, \bar{z}) \psi(z) dx dy.$$

It is known that  $G_f$  is a bounded operator from  $\mathcal{A}^2(\Delta)$  into itself with  $\|G_f\| < 1$  (see [Po1], [Sh1-2], [TT2]). The following relation was proved by the author and Wei [SW]:

$$(5.13) \quad T_h^+ \circ G_f = J \circ T_h^- \circ J,$$

where  $J$  is the operator defined by  $J\phi(z) = \overline{\phi(\bar{z})}$  so that  $J^2 = \text{id}$ ,  $J \circ D = D \circ J$ .

Now let  $\psi \in \mathcal{AD}(\Delta)$  be given. Choose  $\omega \in \mathcal{AD}_0(\Delta)$  so that  $D\omega = -G_f J D\psi$ . By (5.13) it holds that

$$J T_h^- D\psi + T_h^+ D\omega = J T_h^- D\psi - T_h^+ G_f J D\psi = 0.$$

By (5.9) we obtain

$$D(P_h^+ \omega + J P_h^- \psi) = T_h^+ D\omega + J T_h^- D\psi = 0.$$

Then,

$$P P_h(\psi + \bar{\omega}) = P_h^+ \psi + \overline{J P_h^- \psi} + \overline{P_h^+ \omega} + J P_h^- \omega = P_h^+ \psi + J P_h^- \omega + \overline{P_h^+ \omega(0)}.$$

Set  $\phi = P_h^+ \psi + J P_h^- \omega + \overline{P_h^+ \omega(0)}$ . Then  $\phi \in \mathcal{AD}(\Delta)$ , and  $P_{h^{-1}} \phi = \psi + \bar{\omega}$ . Consequently,  $P_{h^{-1}}^+ \phi = \psi$ , and  $P_{h^{-1}}^+$  is surjective. Replacing  $h^{-1}$  with  $h$ , we conclude that  $P_h^+$  is surjective.  $\square$

To proceed, we consider the harmonic conjugation operator  $H$  in the usual sense. Precisely, for a real valued integrable function  $u$  on the unit circle, there exists a unique harmonic function  $v$  on the unit disk with  $v(0) = 0$  such that  $Pu + iv$  is analytic. Then  $Hu = v|_{S^1}$ . When  $u$  is complex valued, set  $Hu = H\Re u + iH\Im u$ . Then,  $\overline{Hu} = H\bar{u}$ , and  $H\phi = -i(\phi - \phi(0))$  when  $\phi$  is holomorphic. We have the following basic result:



**Lemma 5.2.** *For each  $\phi \in \mathcal{AD}(\Delta)$ , it holds that*

$$(5.14) \quad (HP_h + P_h H)\phi = -i(2P_h^+ \phi - P_h^+ \phi(0) - \phi(0)).$$

*Proof.* The proof goes as follows:

$$\begin{aligned} (HP_h + P_h H)\phi &= H(P_h^+ \phi + \overline{JP_h^- \phi}) - iP_h(\phi - \phi(0)) \\ &= -i(P_h^+ \phi - P_h^+ \phi(0)) + i\overline{JP_h^- \phi} - i(P_h^+ \phi + \overline{JP_h^- \phi} - \phi(0)) \\ &= -i(2P_h^+ \phi - P_h^+ \phi(0) - \phi(0)). \quad \square \end{aligned}$$

**Corollary 5.3.** *Let  $v \in H^{\frac{1}{2}}$  be real valued. Then there exists some  $u \in H^{\frac{1}{2}}$  such that  $\|(HP_h + P_h H)u - v\|_{H^{\frac{1}{2}}} = 0$ . Furthermore,  $2\|u\|_{H^{\frac{1}{2}}} \leq \|v\|_{H^{\frac{1}{2}}}$ .*

*Proof.* Set  $\psi = i(v + iHv)/2$ . Then  $P\psi \in \mathcal{AD}(\Delta)$ . By Proposition 5.1, there exists  $\phi \in \mathcal{AD}(\Delta)$  such that  $P_h^+ \phi = P\psi$ . Letting  $u = \Re \phi$ , we obtain by Lemma 5.2 that

$$(HP_h + P_h H)u = \Re(HP_h + P_h H)\phi = \Im(2P_h^+ \phi - (P_h^+ \phi(0) - \phi(0))) = v - \Im(P_h^+ \phi(0) - \phi(0)).$$

Consequently,  $\|(HP_h + P_h H)u - v\|_{H^{\frac{1}{2}}} = 0$ , and by (5.3),

$$4\|u\|_{H^{\frac{1}{2}}}^2 = 2\|\phi\|_{\mathcal{AD}}^2 \leq 2\|P\psi\|_{\mathcal{AD}}^2 = \|v\|_{H^{\frac{1}{2}}}^2. \quad \square$$

**Proof of “if” part:** Suppose  $h$  is an absolutely continuous homeomorphism on the unit circle such that  $\log h' \in H^{\frac{1}{2}}$ . Lemma 3.3 implies that  $h$  is a quasimetric homeomorphism so that Corollary 5.3 may be used. Without loss of generality, we assume  $h(1) = 1$ . Then  $h(e^{i\theta}) = e^{i\phi(\theta)}$ , where  $\phi$  is a strictly increasing and absolutely continuous function on  $[0, 2\pi]$  such that  $\phi(0) = 0$ ,  $\phi(2\pi) = 2\pi$ .

We first assume  $\|\log h'\|_{H^{\frac{1}{2}}}$  is small. By Corollary 5.3, there exists some  $u \in H^{\frac{1}{2}}$  and a real constant  $c_1$  such that

$$(5.15) \quad (HP_h + P_h H)u(\theta) = -H \log \phi'(\theta) - (\phi(\theta) - \theta) + c_1,$$

and  $2\|u\|_{H^{\frac{1}{2}}} \leq \|H \log \phi'(\theta) + (\phi(\theta) - \theta)\|_{H^{\frac{1}{2}}}$  is small. Then there exists a locally univalent analytic function  $f$  on the unit disk with  $f(0) = f'(0) - 1 = 0$  such that for some constant  $c_2$ ,

$$(5.16) \quad \log f'(z) = P(u + iHu)(z) + c_2.$$

Since  $\|\log f'\|_{\mathcal{AD}} = \|u + iHu\|_{H^{\frac{1}{2}}}$  is small, it is well known that  $f$  is univalent in  $\Delta$  and can be extended to a quasiconformal mapping in the whole plane (see [Be]). Consequently,  $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ .

Now we set  $v = P_h u + \log \phi'$ . Then  $\|v\|_{H^{\frac{1}{2}}}$  is small. In fact, when  $\|\log h'\|_{H^{\frac{1}{2}}}$  is small,  $h$  can be extended to a quasiconformal mapping in the unit disk whose Beltrami coefficient  $\mu$  has small norm  $\|\mu\|_\infty$  (see [AZ], [Be]), which in turn implies that  $\|P_h u\|_{H^{\frac{1}{2}}}$  is small (see [HS], [NS]) and so  $\|v\|_{H^{\frac{1}{2}}}$  is also small. By the same reasoning as above, there exists a quasiconformal mapping  $g$  on the whole plane with  $g(\infty) = \infty$  such that  $g$  is conformal in  $\Delta^*$  with  $\log g' \in \mathcal{AD}(\Delta^*)$  and

$$(5.17) \quad \log g'(e^{i\theta}) = v(\theta) - iHv(\theta) + (c_2 + ic_1) = P_h u + \log \phi' - iHP_h u - iH \log \phi' + c_2 + ic_1.$$

Now it follows from (5.15-5.17) that

$$\begin{aligned} P_h \log f' - \log g' &= (P_h u + iP_h H u + c_2) - (P_h u + \log \phi' - iHP_h u - iH \log \phi' + c_2 + ic_1) \\ &= i(P_h H u + HP_h u) - \log \phi' + iH \log \phi' - ic_1 \\ &= -i(H \log \phi'(\theta) + (\phi(\theta) - \theta)) - \log \phi'(\theta) + iH \log \phi'(\theta) \\ &= -\log h'. \end{aligned}$$

Consequently, adding some constant to  $g$  if necessary, it holds that  $g = f \circ h$ . Since  $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ , we conclude that  $h$  belongs to the Weil-Petersson class under the assumption that  $\|\log h'\|_{H^{\frac{1}{2}}}$  is small. It should be pointed out that the above reasoning was inspired by David [Da] in an other setting of BMO theory of the universal Teichmüller space.

When  $\|\log h'\|_{H^{\frac{1}{2}}}$  is not necessarily small, we use an approximation process. Since  $\log h' \in H^{\frac{1}{2}}$ , there exists a sequence  $(u_n)$  of real valued (real) analytic functions such that  $\|u_n - \log \phi'\|_{H^{\frac{1}{2}}} \rightarrow 0$  as  $n \rightarrow \infty$ . Replacing  $u_n$  by  $u_n - a_0(u_n) + a_0(\log \phi')$  if necessary, we may assume that  $a_0(u_n) = a_0(\log \phi')$ . Define  $h_n(e^{i\theta}) = e^{i\phi_n(\theta)}$  by

$$(5.18) \quad \phi_n(\theta) = \frac{2\pi}{\int_0^{2\pi} e^{u_n(t)} dt} \int_0^\theta e^{u_n(t)} dt, \quad \theta \in [0, 2\pi].$$

Then,  $h_n \in \text{WP}(S^1)$  since  $\phi_n$  is a real analytic diffeomorphism.

We first show that  $\|\log h'_n - \log h'\|_{H^{\frac{1}{2}}} \rightarrow 0$  as  $n \rightarrow \infty$ . By our construction,  $\|\log \phi'_n - \log \phi'\|_{H^{\frac{1}{2}}} \rightarrow 0$  as  $n \rightarrow \infty$ . Recall that  $H^{\frac{1}{2}} \subset \text{VMO}$ , and the inclusion map is continuous. Noting that

$$|a_0(e^{u_n}) - 1| = \frac{1}{2\pi} \left| \int_0^{2\pi} (e^{u_n(t)} - e^{\log \phi'}) dt \right| \leq \|e^{u_n(t)} - e^{\log \phi'}\|_1,$$

we conclude by Lemma 3.2 that  $a_0(e^{u_n}) \rightarrow 1$  as  $n \rightarrow \infty$ . Now (5.18) implies that  $\log \phi'_n = u_n - \log a_0(e^{u_n})$ , which implies  $a_0(\log \phi'_n) = a_0(u_n) - \log a_0(e^{u_n}) \rightarrow a_0(\log \phi')$  as  $n \rightarrow \infty$ . By Lemma 3.2 again, we conclude that, for any  $p \geq 1$ ,  $\|\phi'_n - \phi'\|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Now the  $n$ -th ( $n \neq 0$ ) Fourier coefficient of  $\phi_n - \phi$  is

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} (\phi_n(\theta) - \phi(\theta)) e^{-in\theta} d\theta = \frac{1}{2n\pi i} \int_0^{2\pi} (\phi'_n(\theta) - \phi'(\theta)) e^{-in\theta} d\theta,$$

we conclude that

$$\|\phi_n - \phi\|_{H^1} = \sum_{n \neq 0} n^2 |a_n|^2 = \frac{1}{4\pi^2} \sum_{n \neq 0} \left| \int_0^{2\pi} (\phi'_n(\theta) - \phi'(\theta)) e^{-in\theta} d\theta \right|^2 = \|\phi'_n - \phi'\|_2^2,$$

which implies  $\|\phi_n - \phi\|_{H^{\frac{1}{2}}} \leq \|\phi_n - \phi\|_{H^1} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $\|\log h'_n - \log h'\|_{H^{\frac{1}{2}}} \rightarrow 0$  as  $n \rightarrow \infty$ .

Now we consider  $\tilde{h}_n = h_n \circ h^{-1}$ . Then  $\tilde{h}_n$  is absolutely continuous. Noting that

$$\log \tilde{h}'_n = (\log h'_n - \log h') \circ h^{-1} = P_h^{-1}(\log h'_n - \log h'),$$

we find that  $\|\log \tilde{h}'_n\|_{H^{\frac{1}{2}}} \rightarrow 0$  as  $n \rightarrow \infty$ . By what we have proved in the small norm case,  $\tilde{h}_n \in \text{WP}(S^1)$ . Since  $\text{WP}(S^1)$  is a group (in fact a topological group, see [TT2]), we conclude that  $h \in \text{WP}(S^1)$ . Now the proof of Theorem 1.2 is completed.  $\square$

**Remark 5.6:** By means of Theorem 1.2, we can give a new model of the Weil-Petersson Teichmüller space. More precisely, let  $H_{\mathbb{R}}^{\frac{1}{2}}$  denote the subspace of all real-valued functions in  $H^{\frac{1}{2}}$ . By Theorem 1.2,  $\log |h'| \in H_{\mathbb{R}}^{\frac{1}{2}}$  for  $h \in \text{WP}(S^1)$ . Conversely, suppose  $u \in H_{\mathbb{R}}^{\frac{1}{2}}$ . Adding to a constant if necessary, we may assume that  $\int_0^{2\pi} e^{u(t)} dt = 2\pi$ . Set  $h(e^{i\theta}) = e^{i\phi(\theta)}$  by

$$(5.19) \quad \phi(\theta) = \int_0^\theta e^{u(t)} dt, \quad \theta \in [0, 2\pi].$$

Then  $h$  is an absolutely continuous sense-preserving homeomorphism of the unit circle with  $\log |h'| = u$ . By Lemma 3.4 and Theorem 1.2, we get  $h \in \text{WP}(S^1)$ . Consequently, the correspondence  $h \mapsto \log |h'|$  establishes a one-to-one map from  $\text{WP}(S^1)/S^1$  onto  $H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$ . By means of the  $H^{\frac{1}{2}}$  metric, a new metric can be assigned to  $\text{WP}(S^1)/S^1$ . This will be done in the next section (see (6.1)).

## 6. PROOF OF THEOREM 1.4

As stated in the introduction, the universal Teichmüller space has a quasimetric homeomorphism model, namely,  $T = \text{QS}(S^1)/\text{Möb}(S^1)$ . More precisely, there exists a one-to-one mapping from  $T$  onto  $\text{QS}(S^1)/\text{Möb}(S^1)$  which takes a point  $\Phi(\mu)$  to the normalized quasimetric conformal welding corresponding to  $g_\mu^\infty$ . Now  $\mathcal{T} = \text{QS}(S^1)/S^1$  is a fiber space over  $T$  and in fact is a model of the universal Teichmüller curve (see [Ber], [TT2]). Each point in  $\mathcal{T}$  can be considered as a quasimetric homeomorphism which keeps 1 fixed. There exists a one-to-one map  $\Psi$  from  $\mathcal{T}$  onto  $\hat{T}_b$  (an other model of the universal Teichmüller curve) which sends  $h$  to  $\log f'$  under the normalized decomposition  $h = f^{-1} \circ g$ .

Now we consider the Weil-Petersson class. Set  $\mathcal{T}_0 = WP(S^1)/S^1$ . Then  $\Psi$  establishes a bijective map between  $\mathcal{T}_0$  and  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ . As stated in Remark 5.6, a natural metric assigned to  $\mathcal{T}_0$  is the following  $H^{\frac{1}{2}}$  metric:

$$(6.1) \quad d(h_1, h_2) = \|\log |h'_2| - \log |h'_1|\|_{H^{\frac{1}{2}}}, \quad h_1, h_2 \in \mathcal{T}_0.$$

Examining the last step in the proof of Theorem 1.2, we see that the metric is topologically equivalent to the following metric:

$$(6.2) \quad d'(h_1, h_2) = \|\log h'_2 - \log h'_1\|_{H^{\frac{1}{2}}}, \quad h_1, h_2 \in \mathcal{T}_0.$$

Then we have the following result.

**Theorem 6.1.**  $\Psi : (\mathcal{T}_0, d) \rightarrow \hat{T}_b \cap \mathcal{AD}_0(\Delta)$  is a homeomorphism.

*Proof.* We will only sketch the proof. Details can be found in sections 2 and 5. Examining the proof of Theorem 1.2, we find out that  $\|\Psi(h)\|_{\mathcal{AD}}$  is small if  $\|\log h'\|_{H^{\frac{1}{2}}}$  is small. Thus,  $\Psi$  is continuous at the base point  $\text{id}$ . Conversely, if  $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$  has small norm. Let  $h = f^{-1} \circ g$  be the normalized conformal sewing mapping of  $f$ . Since  $S_f = \Lambda(\log f')$  has small norm (2.4), by means of the well-known Ahlfors-Weil section (see [AW]),  $f$  can be extended to a quasiconformal mapping in the whole plane whose complex dilatation  $\mu$  has the form

$$(6.3) \quad \mu(z) = -\frac{1}{2}(|z|^2 - 1)^2 S_f(\bar{z}^{-1}) \bar{z}^{-4}, \quad z \in \Delta^*.$$

Thus,  $\mu \in \mathcal{M}(\Delta^*)$  with small norm  $\|\mu\|_{WP}$ . By means of Lemma 1.5 in [TT2], we have  $f_\mu(\infty) = \infty$ . Since  $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$  has small norm, it must hold that  $f = f_\mu|_\Delta$ . Now let  $w_\mu$  be the unique quasiconformal mapping of  $\Delta^*$  onto itself with Beltrami coefficient  $\mu$  and keeping the points 1 and  $\infty$  fixed. Extending  $w_\mu$  to the unit disk by symmetry, we obtain a quasiconformal mapping  $w_\mu$  in the whole plane with  $w_\mu(0) = 0$ . Then  $g = f_\mu \circ w_\mu^{-1}|_{\Delta^*}$ , and  $h = w_\mu^{-1}|_{S^1}$ . Now lemma 2.2 in [TT2] implies that the Beltrami coefficient  $\nu$  of  $w_\mu^{-1}$  has small norm  $\|\nu\|_{WP}$ . On the other hand, it is easy to see that  $h^{-1} = g^{-1} \circ f$  is the quasisymmetric conformal sewing mapping corresponding to  $rj \circ g \circ j$ , where  $j(z) = \bar{z}^{-1}$  is the standard reflection of the unit circle, and  $r$  is a constant such that  $r(j \circ g \circ j)'(0) = 1$ . Now  $rj \circ g \circ j = rj \circ f_\mu \circ w_\mu^{-1} \circ j|_\Delta$  has the quasiconformal extension  $rj \circ f_\mu \circ w_\mu^{-1} \circ j|_{\Delta^*}$  which keeps the point at infinity fixed, we conclude that  $\log(rj \circ g \circ j)'$  has small norm in  $\mathcal{AD}_0(\Delta)$  since the Beltrami coefficient  $\nu$  of  $w_\mu^{-1}$  has small norm  $\|\nu\|_{WP}$ . Thus,  $\log g'$  has small norm in  $\mathcal{AD}(\Delta^*)$ . It follows from (6.2) that  $\|\log h'\|_{H^{\frac{1}{2}}}$  is small. Consequently,  $\Psi^{-1}$  is continuous at the base point 0.

We will handle the general case by changing a general point to the base point. Let  $h \in \mathcal{T}_0$  be fixed. Consider the map  $R_h$  defined by  $R_h(k) = k \circ h^{-1}$ . Then  $R_h$  is a bijective map from  $\mathcal{T}_0$  onto itself. Noting that

$$(6.4) \quad d'(R_h(k_1), R_h(h_2)) = \|(\log k'_2 - \log k'_1) \circ h^{-1}\|_{H^{\frac{1}{2}}},$$

we conclude that  $R_h$  is a quasi-isometric map from  $\mathcal{T}_0$  onto itself under the  $d'$ -metric. Now let  $w_h$  be a quasiconformal extension of  $h$  to  $\Delta^*$  such that  $w_h$  is quasi-isometric under the Poincaré with Beltrami coefficient  $\mu_h \in \mathcal{M}(\Delta^*)$ . Then  $w_h$  induces a bi-holomorphic isomorphism  $w_h^*$  from  $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$  onto itself which is related to  $R_h$  by  $\Psi \circ w_h^* = \Psi \circ R_h$ . By using the translations  $w_h^*$  and  $R_h$ , we conclude that both  $\Psi : (\mathcal{T}_0, d) \rightarrow \hat{T}_b \cap \mathcal{AD}_0(\Delta)$  and its inverse are continuous at a general point  $h = \Psi(h)$ .  $\square$

**Proof of Theorem 1.4:** The Weil-Petersson Teichmüller space  $T_0 = \text{WP}(S^1)/\text{Möb}(S^1)$  can be considered as a subspace of  $\mathcal{T}_0 = \text{WP}(S^1)/S^1$ , which consists of the normalized conformal sewing mappings corresponding to  $g_\mu^\infty$  with  $\mu \in \mathcal{M}(\Delta^*)$ . By Theorem 6.1,  $\Psi$  establishes a homeomorphism from  $(T_0, d)$  onto  $L_\infty(\mathcal{M}(\Delta^*))$ . On the other hand, Theorem 2.5 implies that  $\Lambda : L_\infty(\mathcal{M}(\Delta^*)) \rightarrow \beta(T_0)$  is a bi-holomorphic isomorphism. This already implies that the metric  $d$  and Weil-Petersson metric induce the same topology on  $T_0 = \text{WP}(S^1)/\text{Möb}(S^1)$ .  $\square$

**Remark 6.2:** By Theorem 6.1,  $\mathcal{T}_0 = \text{WP}(S^1)/S^1$  inherits a complex Hilbert manifold structure from  $\mathcal{AD}_0(\Delta)$  by the homeomorphism  $\Psi : (\mathcal{T}_0, d) \rightarrow \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ . Meanwhile,  $H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$  provides it with a real Hilbert manifold structure by the correspondence  $h \mapsto \log |h'|$  (see Remark 5.6). It is not clear whether these two manifold structures are well compatible with each other. We conjecture that, under the normalized decomposition  $h = f^{-1} \circ g$ , both the bijective map  $\log f' \mapsto \log |h'|$  and its inverse are real analytic.

## REFERENCES

- [Ah] L. V. Ahlfors, *Lectures on Quasiconformal Mapping*, Van Nostrand, 1966.
- [AW] L. V. Ahlfors and G. Weil, *A uniqueness theorem for Beltrami equation*, Proc. Amer. Math. Soc. 13 (1962), 975-978.
- [AG] K. Astala and F. W. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space*, J. Anal. Math. 46 (1986), 16-57.
- [AZ] K. Astala and M. Zinsmeister, *Teichmüller spaces and BMOA*, Math. Ann. 289 (1991), 613-625.
- [Be] J. Becker, *Löwnersche Differentialgleichung*, J. Reine Angew. Math. 255 (1972), 23-43.
- [Ber] L. Bers, *Fiber spaces over Teichmüller spaces*, Acta Math. 130 (1973), 89-126.
- [BA] A. Beurling and L. V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125-142.
- [BR1] M. J. Bowick and S. G. Rajeev, *The holomorphic geometry of closed bosonic string theory and  $\text{Diff } S^1/S^1$* , Nuclear Phys. B 293 (1987), 348-384.
- [BR2] M. J. Bowick and S. G. Rajeev, *String theory as the Kähler geometry of loop space*, Phys. Rev. Lett. 58 (1987), 535-538.
- [Cu] G. Cui, *Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces*, Sci. China Ser. A. 43 (2000), 267-279.
- [Da] G. David, *Courbes corde-arc et espaces de Hardy généralisés*, Ann. Inst. Fourier (Grenoble) 32 (1982), 227-239.
- [DE] A. Douady and C.J. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. 157 (1986), 23-48.
- [EKK] C. J. Earle, I. Kra and S. L. Krushkal, *Holomorphic motions and Teichmüller spaces*, Tran. Amer. Math. Soc. 343 (1994), 927-948.

- [FM] A. Fletcher and V. Markovic, *Quasiconformal Maps and Teichmüller Theory*, Oxford Graduate Texts in Mathematics, vol. 11, Oxford University Press, Oxford, 2007.
- [Fi] A. Figalli, *On flows of  $H^{\frac{3}{2}}$ -vector fields on the circle*, Math. Ann. 347 (2010), 43-57.
- [Ga] F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley-Interscience, New York, 1987.
- [GL] F. P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Mathematical Surveys and Monographs, vol. 76, American Mathematical Society, Providence, RI, 2000.
- [GS] F. P. Gardiner and D. Sullivan, *Symmetric structures on a closed curve*, Amer. J. Math. 114 (1992), 683-736.
- [Gar] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
- [GMR] F. Gay-Balmaz, J. E. Marsden and T. S. Ratiu, *The geometry of the universal Teichmüller space and the Euler-Weil-Petersson equation*, preprint, 2009.
- [HS] Y. Hu and Y. Shen, *On quasisymmetric homeomorphisms*, Israel J. Math. 191 (2012), 209-226.
- [Hu] J. Hubbard, *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics*, vol. 1: Teichmüller Theory, Matrix Editions, Ithaca, NY, 2006.
- [IT] Y. Iwayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer-Verlag, 1992.
- [Ku] S. Kushnarev, *Teichons: Soliton-like geodesics on universal Teichmüller space*, Experiment. Math. 18 (2009), 325-336.
- [Le] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York, 1986.
- [Na] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, Wiley-Interscience, 1988.
- [NS] S. Nag and D. Sullivan, *Teichmüller theory and the universal period mapping via quantum calculus and the  $H^{\frac{1}{2}}$  space on the circle*, Osaka J. Math. 32 (1995), 1-34.
- [NV] S. Nag and A. Verjovsky, *Diff( $S^1$ ) and the Teichmüller space*, Commun. Math. Phys. 130 (1990), 123-138.
- [Ob] B. O'Byrne, *On Finsler geometry and applications to Teichmüller spaces*, Ann. Math. Stud. 66 (1971), 317-328.
- [Pa1] D. Partyka, *The generalized Neumann-Poincaré operator and its spectrum*, Dissertations Math. No. 484, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1997.
- [Pa2] D. Partyka, *Eigenvalues of quasisymmetric automorphisms determined by VMO functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 52 (1998), 121-135.
- [Po1] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, 1975.
- [Po2] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin 1992.
- [Ro] H. Royden, *Automorphisms and isometrics of Teichmüller space*, Ann. Math. Stud. 66 (1971), 369-383.
- [RS] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996.
- [Sh1] Y. Shen, *On Grunsky operator*, Sci. China Ser. A. 50 (2007), 1805-1817.
- [Sh2] Y. Shen, *Faber polynomials with applications to univalent functions with quasiconformal extensions*, Sci. China Ser. A. 52 (2009), 2121-2131.
- [SW] Y. Shen and H. Wei, *Universal Teichmüller space and BMO*, Adv. Math. 234 (2013), 129-148 (2013).
- [TT1] L. Takhtajan and Lee-Peng Teo, *Weil-Petersson geometry of the universal Teichmüller space*, in *Infinite dimensional algebras and quantum integrable systems*, 225-233, Progr. Math. 237, Birkhäuser, Basel, 2005.
- [TT2] L. Takhtajan and Lee-Peng Teo, *Weil-Petersson metric on the universal Teichmüller space*, Mem. Amer. Math. Soc. 183 (2006), no. 861.
- [Te] L. Teo, *The Velling-Kirillov metric on the universal Teichmüller curve*, J. Anal. Math. 93 (2004), 271-308.
- [Tr] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.

- [Wu] C. Wu, *The cross-ratio distortion of integrably asymptotic affine homeomorphism of unit circle*, Sci. China Math. 55 (2012), 625-632.
- [Zh] K. Zhu, *Operator Theory in Function Spaces, Second Edition*, Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, Providence, RI, 2007.
- [Zhu] I. V. Zhuravlev, *A model of the universal Teichmüller space*, Sibirsk. Mat. Zh. 27 (1986), 75-82.
- [Zy] A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge, 1979.

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